REFLECTION OF A STRONG SHOCK WAVE FROM AN ELLIPSOID

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The reflection from an ellipsoid of a strong shock wave (with uniform parameters behind the wave) moving along one axis of the ellipse is considered. Viscosity and thermal conductivity of the gas are not considered. A solution is sought in the vicinity of the critical point using the small parameter method [1]. The nonlinear differential equations for the dimensionless components of the gas velocity in this region are solved by the method of separation of variables with the additional condition of [2]. Analytical expressions are found for the flow parameters, which for the cases of an elliptical cylinder and ellipsoid of revolution coincide with the corresponding expressions obtained previously in [2].

Using the approximations of [3], we write the equations of continuity, and conservation of energy and momentum of [4], describing the gas flow behind a shock wave reflected from an ellipsoid in the vicinity of the critical point ($\theta \ll 1$) in the form

$$
\begin{gathered}
(1+x)\left(\frac{\partial \rho_0}{\partial t_1}+\frac{\partial \rho_0 u_0}{\partial x}\right)+2\rho_0(u_0+v_0)+\rho_0\frac{\partial v_1}{\partial \varphi}=0,\\
\rho_0(1+x)\left(\frac{\partial v_0}{\partial t_1}+u_0\frac{\partial v_0}{\partial x}\right)+\rho_0\left(v_0u_0+v_0^2+v_1\frac{\partial v_0}{\partial \varphi}-v_1^2\right)+\frac{2p^0}{\gamma_3}=0,\\
\rho_0(1+x)\left(\frac{\partial v_1}{\partial t_1}+u_0\frac{\partial v_1}{\partial x}\right)+\rho_0\left(2v_0v_1+v_1u_0+v_1\frac{\partial v_1}{\partial \varphi}\right)+\frac{1}{\gamma_3}\frac{\partial p^0}{\partial \varphi}=0,\\
\rho_0\left(\frac{\partial u_0}{\partial t_1}+u_0\frac{\partial u_0}{\partial x}\right)+\frac{1}{\gamma_3}\frac{\partial p_0}{\partial x}=0,\quad \rho_0\left(\frac{\partial h_0}{\partial t_1}+u_0\frac{\partial h_0}{\partial x}\right)=\frac{\gamma-1}{\gamma}\left(\frac{\partial p_0}{\partial t_1}+u_0\frac{\partial p_0}{\partial x}\right),
\end{gathered}
\tag{1}
$$

where $P/P_3 = p_0(x,\ t_1) + \theta^2 p^0(x,\ t_1,\ \varphi) + \ldots$; $R/R_3 = \rho_0(x,\ t_1) + \theta^2\rho^0(x, t_1, \varphi) + \ldots$; $H/H_3 = h_0(x, t_1) + \theta^2 h^0(x,\ t_1,\ \varphi) + \ldots$; $V_r/a_3 = u_0(x,\ t_1) + \theta^2 u^0(x,\ t_1,\ \varphi) + \ldots$; $V_\theta/a_3 = \theta v_0(x,\ t_1,\ \varphi) + \ldots$; $V_\varphi/a_3 = \theta v_1(x, t_1, \varphi) + \ldots$; $r = c(1+x)$; $t = t_1 c/a_3$; the subscript "3" denotes constant gas parameters behind a shock wave reflected from a plane parallel to the wave front; $a_3^2 = \gamma_3 P_3/R_3$; $H = \gamma(\gamma-1)^{-1}P/R$ is the enthalpy; P is the pressure; R is the density; V_r, V_θ, V_φ are the components of the gas velocity in a spherical coordinate system; t is time; γ is the effective adiabatic index; a, b, and c are the semiaxes of the ellipse (the shock wave moves along the semiaxis c); θ is the angle in the spherical coordinate system.

In the vicinity of the critical point the motion of the reflected shock wave front can be written in the form $F(r_1, \theta, \varphi, t) \equiv r_1 - c - \delta_1(t) - \theta^2\delta_1^0(\varphi, t) - \ldots = 0$. Then the velocity of the reflected wave D can be written in the form [4]

$$D = -|\nabla F|^{-1}F'(t) = \delta_1'(t) \tag{2}$$

(the prime denotes differentiation with respect to the argument within parentheses, or that denoted by the subscript).

Upon passage through the reflected shock wave front the projections of the velocity $-u_2\mathbf{n}_z \approx -u_2\mathbf{n}_r + u_2\theta\mathbf{n}_\theta$ of the gas ahead of the reflected front in the directions of the mutually perpendicular vectors $\mathbf{n}_1 \approx 2\theta r_1^{-1}\delta_1^0\mathbf{n}_r + \mathbf{n}_\theta$ and $\mathbf{n}_2 \approx \theta r_1^{-1}(\delta_1^0)'_\varphi\mathbf{n}_r + \mathbf{n}_\varphi$, lying in a plane tangent to this front, do not change. The component normal to the reflected front $V_{n2} = D - (\mathbf{n}_b, -u_2\mathbf{n}_z)$ ahead of the front is related to the normal component V_n of the gas velocity behind the front by the expression [4]

$$V_{n2}R_2 = V_nR, \tag{3}$$

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where the subscript 2 denotes gas parameters ahead of the reflected front, and $\mathbf{n}_b = |\nabla F|^{-1}\nabla F$ is the normal to the reflected front. Thus, on the reflected front the gas velocity is defined by the equation $\mathbf{V} = V_r\mathbf{n}_r + V_\theta\mathbf{n}_\theta + V_\varphi\mathbf{n}_\varphi = (D - V_n)\,\mathbf{n}_b + \theta u_2\left(1 - 2\delta_1^0/r_1\right)\,\mathbf{n}_1 - \theta r_1^{-1} u_2\left(\delta_1^0\right)'_\varphi \mathbf{n}_2$, whence it follows that

$$V_r = D - V_n,\quad V_\theta = -2(D - V_n)\,\theta\delta_1^0/r_1 + \theta u_2\left(1 - 2\delta_1^0/r_1\right),\qquad V_\varphi = -\theta r_1^{-1}(D - V_n + u_2)\left(\delta_1^0\right)'_\varphi. \tag{4}$$

Then, from Eqs. (2), (3), (4), we obtain

$$u_0 = x_1'(t_1)\left(1 - \rho_{23}/\rho_0\right) - u_{23}\rho_{23}/\rho_0,\quad v_0 = -2\delta_1^0 u_0/r_1 + u_{23}\left(1 - 2\delta_1^0/r_1\right),\qquad v_1 = -(u_0 + u_{23})\,r_1^{-1}\left(\delta_1^0\right)'_\varphi, \tag{5}$$

where $x_1 = \delta_1/c$; $\rho_{23} = R_2/R_3$; $u_{23} = u_2/a_3$.

In the plane φ = const the angle α between the normal to the line of intersection of this plane with the surface of the ellipsoid and the z-axis is connected with the angle θ by the expression $\alpha \approx c^2(a^{-2}\cos^2\varphi + b^{-2}\sin^2\varphi)\,\theta$. In the same plane, the angle β between the normal to the line of intersection of this plane with the reflected front and the z-axis is defined by the relationship [4] $\sin\beta = |\nabla F|^{-1}F'(s) \approx \theta\left[1 - 2\delta_1^0(\varphi, t, a, b, c)/r_1\right]$, where $s = r_1 \sin\theta$. Assuming that in the vicinity of the critical point $\alpha \approx \beta$, we obtain

$$\left(1 - 2\delta_1^0/r_1\right) = c^2\left(a^{-2}\cos^2\varphi + b^{-2}\sin^2\varphi\right). \tag{6}$$

The boundary condition on the body and the initial conditions for t = 0, where the incident shock wave reaches the critical point $\theta = 0$, x = 0 are obtained in the same manner as in [3]:

$$u_0(x = 0) = 0; \tag{7}$$

$$x_1 = 0,\quad x_1'(t_1) = W/a_3 = \varepsilon,\quad p_0 = \rho_0 = h_0 = 1,\quad u_0 = 0,\quad D = W, \tag{8}$$

where W is the velocity of the shock wave reflected from a plane parallel to its front.

We will now assume that for a strong shock wave $\gamma \to 1$ [3]. Then, in analogy to [1, 3], a solution may be sought in the form

$$p_0 = 1 + \varepsilon^2 p + \ldots,\ \rho_0 = 1 + \varepsilon^2\rho + \ldots,\ h_0 = 1 + \varepsilon^2 h + \ldots,\qquad u_0 = \varepsilon u + \ldots,\ v_0 = v\varepsilon^{-1} + \ldots,\qquad v_1 = q\varepsilon^{-1} + \ldots,\ x = \varepsilon^2\xi,\ t_1 = \varepsilon\tau. \tag{9}$$

Now Eq. (1) takes on the form

$$\frac{\partial u}{\partial \xi} + 2v + \frac{\partial q}{\partial \varphi} = 0,\quad \frac{\partial v}{\partial \tau} + u\frac{\partial v}{\partial \xi} + v^2 + q\frac{\partial v}{\partial \varphi} - q^2 = 0,\qquad \frac{\partial q}{\partial \tau} + u\frac{\partial q}{\partial \xi} + q\left(2v + \frac{\partial q}{\partial \varphi}\right) = 0; \tag{10}$$

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} + \frac{1}{\gamma_3}\frac{\partial p}{\partial \xi} = 0; \tag{11}$$

$$\frac{\partial h}{\partial \tau} + u\frac{\partial h}{\partial \xi} = 0. \tag{12}$$

Using the well-known relationship between gas parameters in a strong shock wave, we write the boundary conditions with consideration of Eq. (9), where $\varepsilon^2 = 2(\gamma - 1)(3\gamma - 1)^{-1}$ [3]. Thus, boundary conditions (5) on the reflected front with consideration of Eq. (6), the conditions on the body Eq. (7), and the initial conditions Eq. (8) can be written in the form

$$p/2 = u = h = \xi_1'(\tau) - 1,\quad v = \left(a^{-2}\cos^2\varphi + b^{-2}\sin^2\varphi\right)Ec^2,\qquad q = \sin 2\varphi\left(b^{-2} - a^{-2}\right)Ec^2/2,\quad E = \varepsilon u_{23}; \tag{13}$$

$$u(\xi = 0) = 0; \tag{14}$$

$$\xi_1(\tau = 0) = 0,\quad \xi_1'(\tau = 0) = 1. \tag{15}$$

From Eqs. (10), (13) it follows that at $q = \sin 2\varphi \cdot q_1(\xi, \tau)/2$,

$$\frac{\partial^2 u}{\partial \tau \partial \xi} + u\frac{\partial^2 u}{\partial \xi^2} - \frac{1}{2}\left(\frac{\partial u}{\partial \xi}\right)^2 - \frac{q_1^2}{2} = 0,\quad \frac{\partial q_1}{\partial \tau} + u\frac{\partial q_1}{\partial \xi} - q_1\frac{\partial u}{\partial \xi} = 0; \tag{16}$$

$$\frac{\partial u}{\partial \xi}(\xi = \xi_1) = -Ec^2\left(b^{-2} + a^{-2}\right),\quad q_1(\xi = \xi_1) = Ec^2\left(b^{-2} - a^{-2}\right). \tag{17}$$

We will seek a solution in the form $u = \Phi(\xi)T(\tau)$, $q_1 = \Phi_1(\xi)T_1(\tau)$. Substituting these expressions in Eq. (16), we note that at $T = T_1$ the variables are separable. Then we have

$$T'/T^2 = \lambda, \quad \Phi\Phi_1'/\Phi_1 - \Phi' + \lambda = 0, \quad -\lambda\Phi' - \Phi\Phi'' + (\Phi')^2/2 + \Phi_1^2/2 = 0. \tag{18}$$

From Eq. (18) we obtain an Abel equation of the second sort [5]

$$3\lambda k + k'k\Phi - k^2 - 2\lambda^2 + c_1\Phi = 0. \tag{19}$$

Transforming Eq. (19) into an Abel equation of the first sort, and knowing its special solution, we obtain a solution of Eq. (19) in the form

$$\frac{1}{k} = \frac{\lambda}{2\lambda^2 - c_1\Phi} + \frac{c_2}{2\lambda^2 - c_1\Phi}\left(\frac{c_1\Phi}{2c_2^2 + 2\lambda^2 c_1 c_3 - c_1^2 c_3\Phi}\right)^{1/2}, \tag{20}$$

where $k(\Phi) = \Phi'$; $c_{1,2,3}$ are constants.

From Eq. (18) it follows that

$$T = (c_4 - \lambda\tau)^{-1}, \tag{21}$$

where c_4 is a constant.

In view of the nonlinearity of Eqs. (16) their solution can be represented in the form of sums $\sum_\lambda u_\lambda$ and $\sum_\lambda q_{1\lambda}$ given the condition that

$$\sum_{\lambda, n\neq m} (\pm\lambda_m')\, c_{4m}^{-1} c_{4n}^{-1} M_n = 0, \tag{22}$$
$$\sum_{\lambda, n\neq m} c_{4m}^{-1} c_{4n}^{-1}\, [M_n M_m + (\pm M_{0n})(\pm M_{0m})] = 0,$$

where M takes on either the value 2λ or λ', while M_0 is either zero or λ' $(\lambda' = (-c_2^2 c_1^{-1} c_3^{-1})^{1/2})$.

Condition (22) was obtained from Eq. (16) at $\xi = 0$ and $\tau = 0$. Considering the number of conditions imposed on the eigenvalues λ, we limit ourselves to the case in which the set λ consists of not more than three values: λ_1, $-\lambda_2'$, λ_3' (the remaining combinations do not satisfy the conditions of the problem). The sign of λ corresponds to the sign in the equation

$$\Phi_1 = \pm[(k - 2\lambda)^2 - 2c_1\Phi]^{1/2}, \tag{23}$$

obtained from Eqs. (18), (19). From Eqs. (17), (22) at $\xi_1(\tau = 0) = 0$ we obtain $\omega_1 = 0$, $\omega_2 = -Ec^2/b^2$, $\omega_3 = -Ec^2/a^2$, $\omega \equiv \lambda/c_4$.

Then from Eqs. (14), (20) it follows that

$$k_1 = (2c_{11}\Phi_1 + c_5^2\Phi_1^2)^{1/2}, \quad \Phi_1 = c_{11}(\exp c_5\xi - 1)^2 \exp(-c_5\xi)/(2c_5^2); \tag{24}$$
$$k_2 = \lambda_2' - c_{12}\Phi_2/(2\lambda_2'), \quad \Phi_2 = 2\omega_2^2[1 - \exp(-\nu_2\omega_2^{-1}\xi/2)]\, c_{42}\nu_2^{-1}, \tag{25}$$

where c_5 is a constant; $\nu \equiv c_1/c_4$; $k_1 = k(\lambda_1)$; $k_2 = k(\lambda_2')$.

Expressions for k_3 and Φ_3 are obtained from Eq. (25) by replacing the subscript 2 by 3. From Eqs. (18), (23), (24), and (25) it follows that

$$\Phi_{11} = c_5\Phi_1, \quad \Phi_{12} = -\lambda_2' \exp(-\nu_2\omega_2^{-1}\xi/2), \quad \Phi_{13} = \lambda_3' \exp(-\nu_3\omega_3^{-1}\xi/2), \tag{26}$$

where $\Phi_{11} = \Phi_1(\lambda_1)$; $\Phi_{12} = \Phi_1(\lambda_2')$ ...

Thus, from Eqs. (21), (24), (25), (26) we obtain

$$u = \frac{\nu_1}{2c_5^2}(\exp c_5\xi - 1)^2 \exp(-c_5\xi) + \frac{2\omega_2^2[1 - \exp(-\nu_2\omega_2^{-1}\xi/2)]}{\nu_2(1 - \omega_2\tau)} + \frac{2\omega_3^2[1 - \exp(-\nu_3\omega_3^{-1}\xi/2)]}{\nu_3(1 - \omega_3\tau)}; \tag{27}$$

$$q_1 = \frac{\nu_1}{2c_5}(\exp c_5\xi - 1)^2 \exp(-c_5\xi) - \frac{\omega_2 \exp(-\nu_2\omega_2^{-1}\xi/2)}{1 - \omega_2\tau} + \frac{\omega_3 \exp(-\nu_3\omega_3^{-1}\xi/2)}{1 - \omega_3\tau}. \tag{28}$$

Using Eqs. (16), (17) we find

$$\nu_1 = -2E^2c^4a^{-2}b^{-2}, \quad \nu_2 = 2\omega_2 c_5, \quad \nu_3 = -2\omega_3 c_5, \quad c_5 = -Ec^2(b^{-2} - a^{-2}).$$

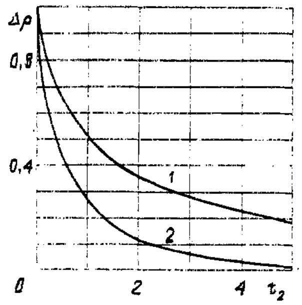

Fig. 1

The value of the dimensionless pressure p is found from Eq. (11) by differentiation of Eq. (27) and subsequent integration. The constant which we obtain in this manner is defined with the aid of the corresponding boundary condition (13). The dimensionless enthalpy is determined from Eq. (12) in the same manner as in [3].

The expressions obtained for the flow parameters at a = b coincide with known expressions [2] for an ellipsoid of revolution, while for $a \to \infty$ or $b \to \infty$, they coincide with corresponding expressions [2] for an elliptical cylinder.

The method employed for solving the problem considered herein may also be used for solution of similar problems in magnetic hydrodynamics [6].

Figure 1 shows the quantity $\Delta p = (P(t) - P(\infty))\,(P(0) - P(\infty))^{-1}$ as a function of dimensionless time $t_2 = U_1 t/c$ (where U_1 is the velocity of the incident shock wave) at the critical point ($\xi = 0$, $\theta = 0$) for a circular cylinder (curve 1) and for a sphere (curve 2). The dependence of Δp upon γ in the range considered is weak.

As follows from Eq. (27) at a = b or $a \to \infty$ $u(\xi = 0, \tau) = u(c^2b^{-2}E\tau) = f(c^2b^{-2}t_2)$. Then it follows from Eq. (11) that $p = p(c^2b^{-2}t_2)$. Thus, for an elliptical cylinder and an ellipsoid of revolution the quantity $\Delta p(c^2b^{-2}t_2)$ is obtained from the curves 1, 2 by a corresponding expansion or compression along the t_2-axis by a factor of c^2/b^2.

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